AVIRAL CLASSES
CREATING SCHOLARS
JEE (ADVANCED), PMT & FOUNDATIONS

WEEKLY TEST TYJ-02 MATHEMATICS SOLUTION 24 NOVEMBER 2019

41. (c) $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}=\lim_{x\to 0}\left(\frac{x^3\cot x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$

$$=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3\times\lim_{x\to 0}\cos x\times\lim_{x\to 0}(1+\cos x)=2$$

42. (d) $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}=\lim_{x\to 0}\frac{2x(e^x-1)}{4.\sin^2\frac{x}{2}}$

$$=2\lim_{x\to 0}\left[\frac{(x/2)^2}{\sin^2\frac{x}{2}}\right]\left(\frac{e^x-1}{x}\right)=2.$$

43. (d) $f(x)=\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$, then

$$\lim_{x\to 0+}f(x)=\lim_{h\to 0}\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right)=\lim_{h\to 0}\frac{e^{1/h}\left(1-\frac{1}{e^{1/h}}\right)}{e^{1/h}\left(1+\frac{1}{e^{1/h}}\right)}=1$$

Similarly $\lim_{x\to 0-}f(x)=-1$. Hence limit does not exist.

44. (c) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\}$

$$=\lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}.\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}.\frac{4}{n^2x^2}\right]$$

$$=\frac{m^2}{n^2}\times 1=\frac{m^2}{n^2}.$$

Aliter : Apply L-Hospital's rule,

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\frac{m\sin mx}{n\sin nx}=\lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx}=\frac{m^2}{n^2}.$$

45. (c) $\lim_{x\to\infty}\sqrt{x}(\sqrt{x+5}-\sqrt{x})\times\frac{(\sqrt{x+5}+\sqrt{x})}{(\sqrt{x+5}+\sqrt{x})}$

$$=\lim_{x\to\infty}\frac{\sqrt{x}(5)}{\sqrt{x}\left(\sqrt{1+\frac{5}{x}}+1\right)}=\frac{5}{2}.$$

46. (b) Apply L-Hospital's rule, $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

$$=\lim_{x\to 0}\frac{\cos x}{2\sqrt{1+\sin x}}+\frac{\cos x}{2\sqrt{1-\sin x}}=\frac{1}{2}+\frac{1}{2}=1.$$

47. (a) $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$

$$= \lim_{\alpha \to \pi/4} \left\{ \frac{\sqrt{2}\left(\sin\alpha . \frac{1}{\sqrt{2}} - \cos\alpha . \frac{1}{\sqrt{2}} \right)}{\left(\alpha - \frac{\pi}{4} \right)} \right\}$$

$$= \sqrt{2} \lim_{\alpha \to \pi/4} \frac{\sin\left(\alpha - \frac{\pi}{4} \right)}{\left(\alpha - \frac{\pi}{4} \right)} = \sqrt{2} \times 1 = \sqrt{2}.$$

Aliter : Apply L-Hospital's rule,

$$\lim_{\alpha \to \pi/4} \frac{\sin\alpha - \cos\alpha}{\alpha - (\pi/4)} = \lim_{\alpha \to \pi/4} \frac{\cos\alpha + \sin\alpha}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

48. (a) $\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{\cos\theta} = \lim_{\theta \to \pi/2} \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \right)} = 0.$

49. (c) $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \left\{ \frac{\frac{2\tan 2x}{2x} - 1}{3 - \frac{\sin x}{x}} \right\} = \frac{1}{2}.$

Aliter : Apply L-Hospital's rule

$$\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \frac{2\sec^2 2x - 1}{3 - \cos x} = \frac{2 - 1}{3 - 1} = \frac{1}{2}.$$

50. (d) $\lim_{x \to 0-} f(x) = \lim_{h \to 0} \frac{0 - h}{h + h^2} = \lim_{h \to 0} \frac{-1}{1 + h} = -1$

and $\lim_{x \to 0+} f(x) = \lim_{h \to 0} \frac{h}{h + h^2} = \lim_{h \to 0} \frac{1}{1 + h} = 1$

Hence limit does not exist.

51. (d) $\lim_{x \to 0} \frac{2\sin 4x \cos 2x}{2\sin x \cos 4x} = \lim_{x \to 0} 4\left(\frac{\sin 4x}{4x} \right)\left(\frac{x}{\sin x} \right)\frac{\cos 2x}{\cos 4x} = 4.$

Aliter : $\lim_{x \to 0} \frac{\frac{2\sin 2x}{2x} + \frac{6\sin 6x}{6x}}{\frac{5\sin 5x}{5x} - \frac{3\sin 3x}{3x}} = \frac{2 + 6}{5 - 3} = 4.$

52. (a) Expand $\sin x$ and then solve.

Aliter : Apply L-Hospital's rule

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x \to 0} \frac{\cos x - 1 + \frac{3x^2}{6}}{5x^4}$$

$$= \lim_{x \to 0} \frac{-\sin x + \frac{6x}{6}}{20x^3} = \lim_{x \to 0} \frac{-\cos x + 1}{60x^2} = \lim_{x \to 0} \frac{\sin x}{120x}$$

$$= \lim_{x \to 0} \frac{\cos x}{120} = \frac{1}{120}.$$

53. (a) Expand $\log(1 + x)$ and then solve.

Aliter : Apply L-Hospital's rule, $\lim_{x \to 0} \left[\frac{x - \log(1 + x)}{x^2} \right]$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{1 + x}}{2x} = \lim_{x \to 0} \frac{1}{2}\left(\frac{1}{1 + x} \right)^2 = \frac{1}{2}.$$

54. (c) $\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{4} = \frac{1}{4}.$

55. (b) $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

$= \lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} \times \frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$

$= \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})} = \frac{2}{3\sqrt{3}}$.

Aliter : Apply L-Hospital's rule.

56. (d) $\pi - 2x = \theta \Rightarrow x = \frac{\pi}{2} - \frac{\theta}{2}$ and as $x \to (\pi/2), \theta \to 0$

Now solve yourself.

57. (a) $\lim_{x\to 0} \frac{4\sin^3 x}{x^3} = 4.$

58. (d) $\lim_{x\to 0} \frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x} = \lim_{x\to 0} \frac{|\sin x|}{x}$

So, $\lim_{x\to 0+} \frac{|\sin x|}{x} = 1$ and $\lim_{x\to 0-} \frac{|\sin x|}{x} = -1$

Hence limit does not exist.

59. (c) $\lim_{x\to \pi/2} \{(1-\sin x)\tan x\} = \lim_{x\to \pi/2} \frac{\sin x - \sin^2 x}{\cos x}$

Apply L-Hospital's rule, we get $\lim_{x\to \pi/2} \frac{\cos x - \sin 2x}{-\sin x} = 0$.

60. (a) $\lim_{x\to 0} \frac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x\to 0} \frac{e^x[e^{\tan x - x} - 1]}{\tan x - x}$

$= \lim_{x\to 0} e^x . \lim_{x\to 0} \frac{e^{\tan x - x} - 1}{\tan x - x} = e^0 \times 1 = 1$.